PMAI LAB • CASE STUDY 01

Renewable Energy Microgrid Planning and Demand Forecasting

Linear Systems • Least Squares • Decision Analysis • Python

Application field
Renewable energy and operations planning

Mathematical focus
Exact systems, matrix rank, least-squares regression, residual analysis

Computational tool
Python with NumPy and Matplotlib

Instructor
Dr. Hira Benish

Plan today's energy mix and use data to prepare for tomorrow's demand.

1. Scenario and Investigation Questions

A campus microgrid must balance supply, cost, environmental impact, and growing demand

A university campus operates a microgrid using three energy sources:

$$\text{solar } (s), \qquad \text{wind } (w), \qquad \text{battery discharge } (b),$$

measured in megawatt-hours (MWh). The daily plan must satisfy:

- total supply of 100 MWh;
- total planning-cost index of 570 units; and
- total lifecycle-impact index of 230 units.

The source coefficients are:

	Solar	Wind	Battery
Supply contribution	1	1	1
Cost index per MWh	6	4	7
Impact index per MWh	2	1	4

Investigation questions

1. How many MWh should be assigned to each source?
2. Is the planning system uniquely solvable?
3. What does recent demand data predict for the next day?
4. Does the planned 100 MWh supply meet the forecast?
5. What operational recommendation follows from the mathematical results?

Recent demand observations

Day t	1	2	3	4	5
Demand y (MWh)	82	87	91	96	104

Learning outcomes

Students will formulate and solve an exact 3×3 system, construct a regression design matrix, fit and evaluate a least-squares line, verify residual orthogonality, use Python, and convert numerical results into an operational decision.

Central challenge
Exact planning determines a feasible energy mix for a stated target. Least squares then asks whether that target remains sufficient when demand is changing.

2. Exact Energy-Mix Model

Translate the operational constraints into a square linear system

The three constraints give

$$\begin{aligned} s + w + b &= 100, \\ 6s + 4w + 7b &= 570, \\ 2s + w + 4b &= 230. \end{aligned}$$

In matrix form,

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 4 & 7 \\ 2 & 1 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} s \\ w \\ b \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 100 \\ 570 \\ 230 \end{pmatrix}.$$

Elimination

Subtract four times the first equation from the second:

$$2s + 3b = 170.$$

Subtract the first equation from the third:

$$s + 3b = 130.$$

Subtracting these two reduced equations gives

$$s = 40.$$

Then

$$40 + 3b = 130 \quad \Longrightarrow \quad b = 30,$$

and

$$w = 100 - 40 - 30 = 30.$$

Therefore,

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 40 \\ 30 \\ 30 \end{pmatrix} \text{ MWh.}}$$

Uniqueness and verification

$$\det(\mathbf{A}) = -3 \neq 0,$$

so **A** is invertible and the solution is unique. Direct substitution gives

$$40 + 30 + 30 = 100,$$

$$6(40) + 4(30) + 7(30) = 570,$$

$$2(40) + 30 + 4(30) = 230.$$

Planning interpretation
The mathematical plan assigns 40% of supply to solar, 30% to wind, and 30% to battery discharge while meeting all three stated targets exactly.

Unit caution
The rows use different units and scales. Scaling equations before numerical work can improve conditioning without changing the physical solution.

3. Least-Squares Demand Forecast

Fit a trend to observations that do not lie exactly on one line

Assume a linear trend

$$\widehat{y} = \beta_0 + \beta_1 t.$$

The design matrix and response vector are

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} 82 \\ 87 \\ 91 \\ 96 \\ 104 \end{pmatrix}.$$

The normal equations are

$$\mathbf{D}^\top \mathbf{D} \widehat{\boldsymbol{\beta}} = \mathbf{D}^\top \boldsymbol{y},$$

or

$$\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{pmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \end{pmatrix} = \begin{pmatrix} 460 \\ 1433 \end{pmatrix}.$$

Solving gives

$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} 76.1 \\ 5.3 \end{pmatrix}, \qquad \boxed{\widehat{y} = 76.1 + 5.3t.}$$

Fitted values and residuals

$$\widehat{\boldsymbol{y}} = \begin{pmatrix} 81.4 \\ 86.7 \\ 92.0 \\ 97.3 \\ 102.6 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} 0.6 \\ 0.3 \\ -1.0 \\ -1.3 \\ 1.4 \end{pmatrix}.$$

The fit measures are

$$\text{SSE} = 5.10, \qquad \text{RMSE} \approx 1.010, \qquad R^2 \approx 0.9822.$$

The normal-equation check is

$$\mathbf{D}^\top \boldsymbol{r} = \mathbf{0}.$$

Day 6 forecast

$$\widehat{y}(6) = 76.1 + 5.3(6) = \boxed{107.9 \text{ MWh}}.$$

> **Interpretation**
> Demand has increased by approximately 5.3 MWh per day over this short observation window. The line fits these five days well, but it remains a short-range trend model.

4. Integrated Decision and Risk Analysis

Combine the exact plan and forecast instead of interpreting either result alone

The exact energy-mix model supplies

$$100 \text{ MWh.}$$

The next-day forecast is

$$107.9 \text{ MWh.}$$

Therefore, the predicted supply gap is

$$\boxed{107.9 - 100 = 7.9 \text{ MWh.}}$$

Operational recommendation

The campus should not rely on the existing 100 MWh plan alone. It should prepare at least 7.9 additional MWh through one or more of the following:

- additional renewable generation or external grid purchase;
- reserved battery capacity;
- scheduled load shifting; or
- demand-response measures.

Why a safety margin is necessary

The RMSE is approximately 1.01 MWh, so the forecast is not exact. A practical plan should include an uncertainty allowance rather than purchasing exactly 7.9 MWh.

Sensitivity questions

- What happens if cloud cover reduces solar availability?
- Is the battery state of charge sufficient for 30 MWh discharge?
- Do cost and lifecycle-impact coefficients remain constant?
- Will the linear demand trend continue beyond Day 6?

Decision statement
Maintain the 40–30–30 source proportions only if source availability and targets remain valid; separately secure additional capacity for the forecast gap and uncertainty.

Responsible limitation
The model omits weather, electricity price variation, equipment limits, charging losses, peak-time constraints, and uncertainty in lifecycle coefficients. The result supports planning, not automatic operation.

Mathematical integration

This decision used an exact system for resource allocation, determinant and rank for uniqueness, least squares for noisy demand, residuals for error analysis, and Python for verification.

5. Python Laboratory

Solve the exact plan, fit the forecast, verify residuals, and visualize the decision

```python
import numpy as np
import matplotlib.pyplot as plt

# Part A: exact energy-mix system
A = np.array([[1., 1., 1.],
              [6., 4., 7.],
              [2., 1., 4.]])
b = np.array([100., 570., 230.])

mix = np.linalg.solve(A, b)
print("Energy mix [solar, wind, battery]:", mix)
print("Determinant:", np.linalg.det(A))
print("Rank:", np.linalg.matrix_rank(A))
print("Verification A @ mix:", A @ mix)

# Part B: least-squares demand forecast
t = np.array([1., 2., 3., 4., 5.])
y = np.array([82., 87., 91., 96., 104.])
D = np.column_stack([np.ones_like(t), t])

beta, residual_sum, rank_D, singular_D = np.linalg.lstsq(
    D, y, rcond=None
)
y_hat = D @ beta
residuals = y - y_hat
sse = residuals @ residuals
rmse = np.sqrt(np.mean(residuals**2))
r_squared = 1 - sse / np.sum((y - y.mean())**2)
normal_check = D.T @ residuals
day6 = np.array([1., 6.]) @ beta
gap = day6 - mix.sum()

print("Trend coefficients:", np.round(beta, 4))
print("SSE, RMSE, R-squared:",
      round(float(sse), 4), round(float(rmse), 4),
      round(float(r_squared), 4))
print("D.T @ residuals:", np.round(normal_check, 10))
print("Day 6 forecast:", round(float(day6), 4))
print("Forecast supply gap:", round(float(gap), 4))

grid = np.linspace(1, 6, 100)
line = beta[0] + beta[1] * grid
plt.scatter(t, y, s=65, color="#1665D8", label="Observed")
plt.plot(grid, line, color="#0C8F8F", label="Least-squares trend")
plt.axhline(mix.sum(), color="#E5A823", ls="--",
            label="Current 100 MWh plan")
plt.scatter([6], [day6], color="#E5A823", s=75,
            label="Day 6 forecast")
plt.xlabel("Day")
plt.ylabel("Demand (MWh)")
plt.title("Microgrid demand forecast")
plt.legend()
plt.grid(alpha=0.25)
plt.tight_layout()
plt.show()
```

Selected output

```
Energy mix [solar, wind, battery]: [40. 30. 30.]
Determinant: -3.0    Rank: 3
Trend coefficients: [76.1 5.3]
SSE, RMSE, R-squared: 5.1 1.01 0.9822
D.T @ residuals: [0. 0.]
Day 6 forecast: 107.9    Forecast supply gap: 7.9
```

6. Student Investigation Tasks

Extend the case study through computation, interpretation, and sensitivity analysis

Part A: mathematical verification

1. Compute $\det(\mathbf{A})$ and explain why the energy mix is unique.
2. Verify all three planning constraints using $(s, w, b) = (40, 30, 30)$.
3. Calculate the percentage contribution of each source.

Part B: changed operating scale

Suppose demand, cost target, and impact target all rise by 10%:

$$\boldsymbol{b}_{\text{new}} = \begin{pmatrix} 110 \\ 627 \\ 253 \end{pmatrix}.$$

4. Solve $\mathbf{A}\boldsymbol{x}_{\text{new}} = \boldsymbol{b}_{\text{new}}$.
5. Explain why the new solution is a scaled version of the original.

Part C: update the forecast

The actual Day 6 demand is later observed to be 109 MWh.

6. Refit the trend using all six days.
7. Predict Day 7 demand.
8. Compare the new slope with the original 5.3 MWh-per-day slope.

Part D: decision reflection

9. Explain why R^2 alone is not enough to guarantee a safe energy plan.
10. Identify three additional variables that a realistic microgrid model should include.
11. Suggest how the planning model could be reformulated as an optimization problem.

Submission expectation
Show the matrix model, calculations, Python output, one graph, an operational recommendation, and at least two limitations. Every numerical answer must be interpreted in its physical units.

Extension
Advanced students may add inequality constraints such as maximum solar availability, minimum battery reserve, and nonnegative source allocations, then solve using linear programming.

7. Answers, Takeaways, and Extension

Brief answers

Task	Answer
1	$\det(\mathbf{A}) = -3 \neq 0$, so **A** is invertible and the solution is unique.
2	The supply, cost, and impact totals are 100, 570, and 230.
3	Solar 40%, wind 30%, battery 30%.
4	$\boldsymbol{x}_{\text{new}} = (44, 33, 33)^{\mathsf{T}}$ MWh.
5	Every right-hand-side target was multiplied by 1.1, so linearity gives $\boldsymbol{x}_{\text{new}} = 1.1\boldsymbol{x}$.
6	Updated trend: $\widehat{y} \approx 75.7333 + 5.4571t$.
7	Day 7 prediction ≈ 113.9333 MWh.
8	The slope increases from 5.3 to approximately 5.4571 MWh per day.

Model-development pathway

- **Current model:** exact equality constraints plus a linear trend.
- **Next model:** add bounds, source availability, efficiency, and uncertainty.
- **Advanced model:** minimize cost and environmental impact subject to demand, reliability, and storage constraints.

Key takeaways

- Linear systems provide transparent resource-allocation models.
- Determinant and rank establish whether the allocation is unique.
- Least squares extracts a trend from noisy observations.
- Residuals and RMSE quantify what the fitted line does not explain.
- A useful decision combines mathematical outputs with uncertainty and operational context.

Final conclusion

The exact mix is $(40, 30, 30)^{\mathsf{T}}$ MWh, while the Day 6 forecast is 107.9 MWh. The campus should plan additional capacity beyond the current 100 MWh target and include a safety margin.

Key terms: linear system • invertibility • determinant • rank • design matrix • least squares • residual • RMSE • R^2 • forecasting • sensitivity