PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 01 • Linear Systems and Mathematical Modelling

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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A practical first step from real situations to mathematical decisions.

1. Why Linear Systems Matter

From several conditions to one mathematically consistent decision

Many practical problems contain several unknown quantities connected by several conditions. A linear system organizes those conditions so that we can determine whether a solution exists, whether it is unique, and what it means in the original situation.

Situation → **Variables** → **Equations** → **Matrix form** → **Solution** → **Meaning**

Learning outcomes

By the end of this lecture, students should be able to:

- translate a verbal situation into variables and linear equations;
- represent a system as $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ and as an augmented matrix $[\mathbf{A} \mid \boldsymbol{b}]$;
- solve systems using Gaussian and Gauss-Jordan elimination;
- classify systems as unique, infinitely many, or inconsistent;
- use Python to solve and verify an applied linear system; and
- interpret numerical output in the language of the application.

Prerequisite knowledge

Algebraic manipulation of equations, basic matrix notation, and substitution or elimination.

Central idea
Solving a system is not only about finding numbers. It also tests whether the model's conditions are compatible and whether the available information is sufficient for a definite decision.

A first model

Suppose a service sells standard and premium packages. Let x and y denote the numbers sold. Then

$$x + y = 40, \qquad 20x + 35y = 1100$$

form a linear system. The first equation represents total quantity; the second represents total revenue.

2. Mathematical Language and Notation

A linear equation in variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are known scalars. A linear system is a collection of such equations that must be satisfied simultaneously.

Matrices and vectors are written differently

Throughout these notes, uppercase bold symbols and square brackets represent matrices, while lowercase bold symbols and parenthesized columns represent vectors:

$$\underbrace{\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}}_{\text{matrix: square brackets}} \qquad \underbrace{\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{vector: parentheses}} \qquad \underbrace{\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{\text{vector: parentheses}}.$$

For example,

$$\left.\begin{aligned} x + 2y - z &= 3, \\ 2x - y + z &= 4 \end{aligned}\right\} \iff \mathbf{A}\boldsymbol{x} = \boldsymbol{b} \iff \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 1 & 4 \end{array}\right].$$

Notation rule
$\mathbf{A}$ is a matrix, $\boldsymbol{x}$ and $\boldsymbol{b}$ are column vectors, and $[\mathbf{A} \mid \boldsymbol{b}]$ is an augmented matrix. Bold notation and bracket shape allow the objects to be recognized immediately.

Elementary row operations

$$\begin{aligned} \text{Interchange rows:} \quad & R_i \leftrightarrow R_j, \\ \text{Scale a row:} \quad & R_i \leftarrow cR_i, \quad c \neq 0, \\ \text{Replace a row:} \quad & R_i \leftarrow R_i + cR_j. \end{aligned}$$

Why are these operations valid?
Each operation produces an equivalent system with exactly the same solution set. The equations become simpler without changing what counts as a solution.

Gaussian versus Gauss-Jordan elimination

Method	Stopping point	How the solution is obtained
Gaussian elimination	Row-echelon form	Use back-substitution
Gauss-Jordan elimination	Reduced row-echelon form	Read the solution directly

3. Worked Example: A Unique Solution

Use Gaussian elimination and interpret every row operation

Solve the system

$$\begin{aligned} x + y + z &= 6, \\ 2x - y + z &= 3, \\ x + 2y - z &= 2. \end{aligned}$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 2 \end{array}\right].$$

Step 2: Eliminate x from rows 2 and 3

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - R_1,$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -4 \end{array}\right].$$

Step 3: Swap rows 2 and 3, then eliminate y from row 3

$$R_2 \leftrightarrow R_3, \qquad R_3 \leftarrow R_3 + 3R_2,$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -7 & -21 \end{array}\right].$$

Step 4: Back-substitute

$$-7z = -21 \Rightarrow z = 3, \qquad y - 2(3) = -4 \Rightarrow y = 2, \qquad x + 2 + 3 = 6 \Rightarrow x = 1.$$

Therefore,

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}.$$

Verification
Substituting $(x, y, z) = (1, 2, 3)$ into all three original equations satisfies every equation.

Geometric meaning. Each equation represents a plane in $\mathbb{R}^3$. A unique solution means that the three planes meet at exactly one point, $(1, 2, 3)$.

4. When Does a System Have a Solution?

Row reduction reveals the number of solutions. The key evidence is the position of pivots and whether a contradiction appears.

Classification	Row-reduction evidence	Geometric picture
Unique solution	A pivot in every variable column	Planes meet at one point
Infinitely many solutions	At least one free variable; no contradiction	Planes share a line or plane
No solution	A row $[0 \ \cdots \ 0 \mid c]$, $c \neq 0$	Parallel or incompatible planes

Example A: Infinitely many solutions

$$\begin{aligned} x + y + z &= 4, \\ 2x + 2y + 2z &= 8, \\ x - y + z &= 2. \end{aligned}$$

The second equation repeats the first. Subtracting the third equation from the first gives $2y = 2$, so $y = 1$. Then $x + z = 3$. Let $z = t$. The complete solution is

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 3-t \\ 1 \\ t \end{pmatrix}, \quad t \in \mathbb{R}.}$$

The parameter t represents one degree of freedom.

Example B: No solution

$$x + y + z = 4, \qquad 2x + 2y + 2z = 9.$$

Doubling the first left-hand side gives the second left-hand side, but doubling 4 gives 8, not 9. Row reduction produces

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right] \iff 0 = 1,$$

which is a contradiction.

Rank test

The system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent when

$$\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \mid \boldsymbol{b}]).$$

If the common rank equals the number of variables, the solution is unique. If it is smaller, free variables produce infinitely many solutions.

5. Application Case Study: Planning City Shuttle Trips

A realistic model using totals, capacity, and operating resources

A city-university shuttle service plans daily trips on three routes. Route A uses a 30-seat vehicle and consumes 6 fuel units per trip. Route B uses a 40-seat vehicle and consumes 5 fuel units. Route C uses a 50-seat vehicle and consumes 8 fuel units.

The daily plan must provide 18 trips, 720 passenger places, and use 114 fuel units.

Step 1: Define variables

Let x, y, and z denote the numbers of daily trips on Routes A, B, and C, respectively. Thus,

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Step 2: Form the equations

$$\begin{aligned} x + y + z &= 18 && \text{(total trips)}, \\ 30x + 40y + 50z &= 720 && \text{(passenger places)}, \\ 6x + 5y + 8z &= 114 && \text{(fuel units)}. \end{aligned}$$

Step 3: Write the matrix model

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 30 & 40 & 50 \\ 6 & 5 & 8 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} 18 \\ 720 \\ 114 \end{pmatrix}}_{\boldsymbol{b}}.$$

Step 4: Eliminate and solve

$$\begin{aligned} R_2 \leftarrow R_2 - 30R_1 : &\quad 10y + 20z = 180, \\ R_3 \leftarrow R_3 - 6R_1 : &\quad -y + 2z = 6. \end{aligned}$$

From $-y + 2z = 6$, we obtain $y = 2z - 6$. Substitution into $10y + 20z = 180$ gives $40z = 240$, so $z = 6$. Therefore $y = 6$ and $x = 18 - 6 - 6 = 6$.

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}.}$$

Decision
Operate 6 trips on each route. This supplies 18 trips, 720 passenger places, and uses exactly 114 fuel units. The mathematical answer becomes an operational decision only after these checks.

6. Python Laboratory

Solve, verify, and interpret the shuttle-planning model

The following code runs from top to bottom in Google Colab. It implements the mathematical model and verifies the solution.

```
import numpy as np

# Coefficient matrix: columns correspond to Routes A, B and C
A = np.array([
    [1,  1,  1],
    [30, 40, 50],
    [6,  5,  8]
], dtype=float)

# Required totals: trips, passenger places and fuel units
b = np.array([18, 720, 114], dtype=float)

# Solve A x = b
x = np.linalg.solve(A, b)

# Diagnostics and verification
det_A = np.linalg.det(A)
rank_A = np.linalg.matrix_rank(A)
check = A @ x
residual = b - check

print("Trips on Routes A, B and C:", x)
print("det(A):", round(det_A, 6))
print("rank(A):", rank_A)
print("A @ x:", check)
print("Residual:", residual)
```

Expected output

```
Trips on Routes A, B and C: [6. 6. 6.]
det(A): 40.0
rank(A): 3
A @ x: [ 18. 720. 114.]
Residual: [0. 0. 0.]
```

Mathematical interpretation

- The array $[6,6,6]$ represents the vector $\boldsymbol{x} = \begin{pmatrix} 6 & 6 & 6 \end{pmatrix}^{\mathsf{T}}$.
- $\det(\mathbf{A}) = 40 \neq 0$, so $\mathbf{A}$ is invertible and the solution is unique.
- $\operatorname{rank}(\mathbf{A}) = 3$, equal to the number of variables.
- The calculation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ verifies all original constraints simultaneously.
- The residual vector $\boldsymbol{b} - \mathbf{A}\boldsymbol{x} = \mathbf{0}$ shows no discrepancy.

Python decision rule
Use `np.linalg.solve(A, b)` for a square nonsingular system. Do not calculate $\mathbf{A}^{-1}$ merely to solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$; a direct solver is clearer and generally more numerically reliable.

7. Concept Check and Common Misconceptions

Common mistakes

Mistake	Why it fails	Better practice
Changing only one entry in a row	A row operation acts on the entire equation	Apply it to every coefficient and the right-hand side
Dividing by zero	It is not an allowable row operation	Swap rows or choose a different pivot
Stopping after finding one variable	Every variable and constraint must be satisfied	Back-substitute and verify
Assuming every system is solvable	Conditions may contradict one another	Look for $[0\ \cdots\ 0 \mid c]$, $c \neq 0$
Reporting numbers without units	The answer loses contextual meaning	State variables, units, and feasibility

Three concept-check MCQs

1. Which augmented row proves that a system is inconsistent?

A. $[0\ 1\ 3 \mid 2]$

B. $[0\ 0\ 0 \mid 0]$

C. $[0\ 0\ 0 \mid 5]$

D. $[1\ 0\ 0 \mid 4]$

2. A consistent system with three variables has rank 2. What follows?

A. It has no solution.

B. It has a unique solution.

C. It has infinitely many solutions.

D. Its determinant is 1.

3. What does the Python expression `A @ x` calculate?

A. A determinant

B. A matrix-vector product

C. A matrix inverse

D. A row swap

MCQ answers
1-C, 2-C, 3-B

One-minute reflection

If Python returns a numerical vector x, what mathematical and contextual checks should be performed before accepting it as a decision?

8. Practice Exercises

Progress from technique to modelling and interpretation

Foundation

1. Write the augmented matrix for $2x + y = 7$ and $x - 3y = -5$.
2. Solve by Gaussian elimination: $x + y = 7$ and $2x - y = 5$.
3. Classify the system $x + 2y = 4$ and $2x + 4y = 8$.

Application

4. A workshop produces basic and advanced kits. A total of 50 kits are made. Basic kits require 2 labour hours and advanced kits require 5 hours. If 160 labour hours are used, determine the number of each kit.
5. Use row reduction to decide whether the system is consistent. Explain what the conclusion would mean if the equations represented business constraints:

$$x + y + z = 5, \qquad 2x + 2y + 2z = 10, \qquad x - y = 1.$$

6. Modify the shuttle Python code by changing the fuel budget from 114 to 120. Solve the new system and decide whether the result is operationally meaningful when trip counts must be nonnegative integers.

Mathematical challenge

7. Find the value of k for which

$$x + y = 2, \qquad 2x + ky = 4$$

   has infinitely many solutions. What happens for other values of k?
8. Explain why a square system with $\det(\mathbf{A}) \neq 0$ must have a unique solution. You may use invertibility rather than a long formal proof.

Apply and Explore: Design your own model
Choose a realistic allocation problem with two or three unknowns. Define the variables, form an equal number of independent linear equations, solve by hand, solve with Python, and explain whether the answer is feasible. Suggested contexts include class scheduling, event budgeting, food production, energy distribution, and transport planning.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem

A relief organization prepares three types of supply boxes: food (F), medical (M), and mixed (X). It must prepare 30 boxes. Food, medical, and mixed boxes require 2, 1, and 2 volunteer-hours respectively, with 52 volunteer-hours available. Their transport weights are 10 kg, 8 kg, and 12 kg, with a total target of 304 kg.

Define the variables, formulate the system, solve it using Gaussian elimination, verify it using Python, and interpret the result with its units and feasibility conditions.

Brief answers to practice

Exercise	Brief answer
1	$\left[\begin{array}{cc\|c} 2 & 1 & 7 \\ 1 & -3 & -5 \end{array}\right]$
2	$x = 4$, $y = 3$
3	Infinitely many solutions
4	30 basic kits and 20 advanced kits
5	Consistent; infinitely many solutions
6	$\boldsymbol{x} = \begin{pmatrix} 7.5 & 3 & 7.5 \end{pmatrix}^{\mathsf{T}}$; not feasible when trips must be integers
7	$k = 2$ gives infinitely many solutions; $k \neq 2$ gives a unique solution
Challenge scenario	$\begin{pmatrix} F & M & X \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 12 & 8 & 10 \end{pmatrix}^{\mathsf{T}}$ boxes

Lecture summary

- A linear system expresses several conditions that must hold simultaneously.
- Matrix form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ separates coefficients, unknowns, and required totals.
- Row operations preserve the solution set and expose the system's structure.
- Pivots and contradictions determine whether solutions are unique, infinite, or absent.
- Python accelerates computation, but modelling, verification, and interpretation remain mathematical responsibilities.

Research and career connection

Large linear systems appear in scientific computing, optimization, network analysis, computer graphics, economics, engineering simulation, and machine learning. Later lectures will extend these ideas through rank, projections, eigenvalue methods, least squares, and matrix decompositions.

Key terms: linear equation • system • coefficient matrix • augmented matrix • pivot • row-echelon form • free variable • consistency • residual • mathematical model

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy documentation.