PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 02 • Matrices, Determinants, and Inverses

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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Matrices organize relationships; determinants and inverses reveal whether those relationships can be reversed.

1. Why Matrices Matter

A compact language for organizing data, equations, networks, and transformations

A matrix is a rectangular arrangement of numbers with meaning determined by its context. Rows may represent products, locations, or observations; columns may represent features, resources, or time periods. Matrix operations allow many related calculations to be performed as one structured computation.

Connection with Lecture 1

In Lecture 1, a system of equations was written as $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. We now study the matrix **A** itself: how matrices are combined, multiplied, measured by determinants, and reversed through inverses.

Learning outcomes

By the end of this lecture, students should be able to:

- identify the order and important types of matrices;
- perform matrix addition, scalar multiplication, multiplication, and transpose;
- explain why matrix multiplication depends on dimensions and order;
- calculate and interpret determinants of 2×2 and 3×3 matrices;
- determine whether a square matrix is invertible;
- calculate a 2×2 inverse and use it to solve a system; and
- formulate and solve an economic input-output model using Python.

Notation standard

Matrices use uppercase bold symbols and square brackets; vectors use lowercase bold symbols and parenthesized columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

If **A** has m rows and n columns, then $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Central idea
Dimensions tell us whether an operation is defined. The determinant tells us whether a square matrix loses information. An inverse exists only when the matrix preserves enough information to reverse its action.

Where matrices appear

Data tables	Digital images	Networks	Economic models	AI systems

2. Matrix Language, Types, and Basic Operations

For

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 5 \end{bmatrix},$$

A has order 2×3. The entry $a_{23} = 5$ lies in row 2, column 3.

Common matrix types

Type	Defining feature	Example
Row matrix	One row	$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
Column matrix	One column	$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$
Square matrix	Same number of rows and columns	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
Diagonal matrix	Off-diagonal entries are zero	$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$
Identity matrix	Diagonal entries are 1	$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Symmetric matrix	$\mathbf{A}^\top = \mathbf{A}$	$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Equality, addition, and scalar multiplication

Matrices are equal only when they have the same order and corresponding entries are equal. Addition is also entrywise and requires equal dimensions.

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ -2 & 0 \end{bmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 4 \end{bmatrix}, \qquad 2\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 6 & 8 \end{bmatrix}.$$

Transpose

The transpose interchanges rows and columns:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \implies \mathbf{A}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Important properties include

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top, \qquad (\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top.$$

Interpretation
Transposing a data matrix exchanges the roles of observations and features. The numbers do not change, but their structural meaning does.

3. Matrix Multiplication

Rows of the first matrix interact with columns of the second

The product **AB** is defined when the number of columns of **A** equals the number of rows of **B**:

$$\underbrace{\mathbf{A}}_{m\times n}\underbrace{\mathbf{B}}_{n\times p} = \underbrace{\mathbf{AB}}_{m\times p}.$$

The entry in row i, column j is a row-column dot product:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Worked example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 4 & 2 \end{bmatrix}.$$

Because **A** is 2×3 and **B** is 3×2, the product **AB** is 2×2.

$$\mathbf{AB} = \begin{bmatrix} 1(2)+2(0)+0(4) & 1(1)+2(-1)+0(2) \\ (-1)(2)+3(0)+1(4) & (-1)(1)+3(-1)+1(2) \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}}.$$

Order matters
Matrix multiplication is generally not commutative: $\mathbf{AB} \neq \mathbf{BA}$. Even when both products exist, they may have different orders or different entries.

A data interpretation

Suppose the rows of

$$\mathbf{Q} = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$$

contain two students' scores on three assessment components, while

$$\boldsymbol{w} = \begin{pmatrix} 0.30 \\ 0.30 \\ 0.40 \end{pmatrix}$$

contains their weights. Then

$$\mathbf{Q}\boldsymbol{w} = \begin{pmatrix} 4(0.30)+3(0.30)+5(0.40) \\ 2(0.30)+5(0.30)+4(0.40) \end{pmatrix} = \begin{pmatrix} 4.1 \\ 3.7 \end{pmatrix}.$$

The matrix-vector product calculates both weighted scores simultaneously.

Dimension check before calculation

Write the dimensions above each factor. If the inner dimensions do not match, the product is undefined. This simple check prevents many errors.

4. Determinants

A scalar that detects invertibility and measures signed scaling

The determinant is defined only for square matrices. For

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \det(\mathbf{A}) = ad - bc.$$

Geometric meaning

For a 2×2 matrix, $|\det(\mathbf{A})|$ is the area-scaling factor. If $\det(\mathbf{A}) = 0$, a two-dimensional region is collapsed into a line or point, so information is lost and the action cannot be reversed.

Worked 3×3 example

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Expanding along the first row,

$$\begin{aligned} \det(\mathbf{A}) &= 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \\ &= 2(6-2) - (2-0) = 8 - 2 = \boxed{6}. \end{aligned}$$

Since $\det(\mathbf{A}) \neq 0$, the matrix is invertible.

Essential determinant properties

Operation or structure	Effect on the determinant
Interchange two rows	Changes the sign
Multiply one row by c	Multiplies the determinant by c
Add a multiple of one row to another	No change
Triangular matrix	Product of diagonal entries
Product $\mathbf{AB}$	$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
Transpose	$\det(\mathbf{A}^\top) = \det(\mathbf{A})$

Determinant as a decision test

$$\det(\mathbf{A}) \neq 0 \iff \mathbf{A}^{-1} \text{ exists} \iff \mathbf{A}\boldsymbol{x} = \boldsymbol{b} \text{ has a unique solution for every } \boldsymbol{b}.$$

The determinant is therefore more than a calculation: it is a structural test.

5. Inverse Matrices

Reversing the action of a nonsingular square matrix

A square matrix $\mathbf{A}$ is invertible if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

For a 2×2 matrix,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad ad - bc \neq 0.$$

Worked example

Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

First,

$$\det(\mathbf{A}) = 3(1) - 1(2) = 1.$$

Therefore,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

Verification gives

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Solving a system with the inverse

Suppose

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \boldsymbol{b} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}.$$

Multiplying by $\mathbf{A}^{-1}$ gives

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{pmatrix} 11 \\ 7 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}.$$

Row-reduction method for larger matrices

The inverse may also be found by reducing the augmented matrix

$$[\mathbf{A} \mid \mathbf{I}] \longrightarrow [\mathbf{I} \mid \mathbf{A}^{-1}].$$

If the left side cannot be reduced to $\mathbf{I}$, the matrix is singular and has no inverse.

Computational caution
Although inverses are conceptually important, numerical software normally solves $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ directly. Explicitly forming $\mathbf{A}^{-1}$ can be slower and less stable.

6. Application Case Study: A Three-Sector Economy

Using matrices to include both final demand and inter-sector demand

Consider three connected sectors: Agriculture, Manufacturing, and Services. Each sector uses some output from all three sectors in order to produce its own output.

Let

$$\boldsymbol{x} = \begin{pmatrix} x_A \\ x_M \\ x_S \end{pmatrix}$$

represent total production, and let $\boldsymbol{d}$ represent demand from consumers, government, exports, and other final users.

The consumption matrix

$$\mathbf{C} = \begin{bmatrix} 0.20 & 0.10 & 0.05 \\ 0.10 & 0.15 & 0.10 \\ 0.05 & 0.10 & 0.20 \end{bmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} 66 \\ 46 \\ 83 \end{pmatrix}.$$

The entry c_{ij} records the amount supplied by sector i that is required per unit of output from sector j.

Forming the model

Total production must cover internal demand $\mathbf{C}\boldsymbol{x}$ and final demand $\boldsymbol{d}$:

$$\boldsymbol{x} = \mathbf{C}\boldsymbol{x} + \boldsymbol{d}.$$

Rearranging,

$$(\mathbf{I} - \mathbf{C})\boldsymbol{x} = \boldsymbol{d}, \qquad \boldsymbol{x} = (\mathbf{I} - \mathbf{C})^{-1}\boldsymbol{d}.$$

Here,

$$\mathbf{I} - \mathbf{C} = \begin{bmatrix} 0.80 & -0.10 & -0.05 \\ -0.10 & 0.85 & -0.10 \\ -0.05 & -0.10 & 0.80 \end{bmatrix}, \qquad \det(\mathbf{I} - \mathbf{C}) = 0.524875 \neq 0.$$

Therefore the model has a unique solution:

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 100 \\ 80 \\ 120 \end{pmatrix}}.$$

Interpretation and verification

Internal sector demand is

$$\mathbf{C}\boldsymbol{x} = \begin{pmatrix} 34 \\ 34 \\ 37 \end{pmatrix}.$$

Adding final demand gives

$$\mathbf{C}\boldsymbol{x} + \boldsymbol{d} = \begin{pmatrix} 34 \\ 34 \\ 37 \end{pmatrix} + \begin{pmatrix} 66 \\ 46 \\ 83 \end{pmatrix} = \begin{pmatrix} 100 \\ 80 \\ 120 \end{pmatrix} = \boldsymbol{x}.$$

Planning decision
To satisfy both inter-sector requirements and final demand, the economy must produce 100 units of Agriculture, 80 units of Manufacturing, and 120 units of Services. Producing only the final-demand vector would leave the sectors without the inputs needed for production.

7. Python Laboratory

Solve and verify the three-sector input-output model

```
import numpy as np

# Consumption matrix: Agriculture, Manufacturing, Services
C = np.array([
    [0.20, 0.10, 0.05],
    [0.10, 0.15, 0.10],
    [0.05, 0.10, 0.20]
], dtype=float)

# Final-demand vector
d = np.array([66, 46, 83], dtype=float)

I = np.eye(3)
B = I - C

# Solve (I - C)x = d
x = np.linalg.solve(B, d)

# Structural and contextual checks
det_B = np.linalg.det(B)
internal_demand = C @ x
check = internal_demand + d
residual = x - check

print("det(I - C):", round(det_B, 6))
print("Total production:", x)
print("Internal demand:", internal_demand)
print("Internal + final demand:", check)
print("Residual:", residual)
```

Expected output

```
det(I - C): 0.524875
Total production: [100.  80. 120.]
Internal demand: [34. 34. 37.]
Internal + final demand: [100.  80. 120.]
Residual: [0. 0. 0.]
```

Mathematical interpretation

- $\det(\mathbf{I} - \mathbf{C}) \neq 0$, so the production requirements have a unique solution.
- The vector $\boldsymbol{x} = \begin{pmatrix} 100 & 80 & 120 \end{pmatrix}^{\mathsf{T}}$ is total production, not merely final demand.
- The residual $\boldsymbol{x} - (\mathbf{C}\boldsymbol{x} + \boldsymbol{d}) = \mathbf{0}$ verifies the economic balance equation.

Code notation versus mathematical notation
NumPy uses square brackets to store both one-dimensional arrays and two-dimensional arrays. In written mathematics, this lecture distinguishes them: matrices use square brackets, while vectors are bold parenthesized columns. The code syntax is a storage convention, not a change in the mathematics.

8. Concept Check and Practice

Common misconceptions

Misconception	Why it fails	Correct practice
$\mathbf{AB} = \mathbf{BA}$	Matrix multiplication is not generally commutative	Check both order and dimensions
Every matrix has an inverse	Only nonsingular square matrices are invertible	Test $\det(\mathbf{A}) \neq 0$
$\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$	Determinants are not additive	Use valid determinant properties only
$\mathbf{A}^{-1} = 1/\mathbf{A}$	Division by a matrix is not entrywise division	Use the inverse definition or formula
Ignoring units in applications	Entries have contextual meanings	Label rows, columns, and units

Three concept-check MCQs

1. If $\mathbf{A}$ is 3×4 and $\mathbf{B}$ is 4×2, then $\mathbf{AB}$ has order:

A. 3×2

B. 4×4

C. 2×3

D. undefined

2. If $\det(\mathbf{A}) = 0$, then:

A. $\mathbf{A} = \mathbf{I}$

B. $\mathbf{A}^{-1}$ does not exist

C. $\mathbf{A}$ is diagonal

D. $\mathbf{A}^\top = 0$

3. In the model $\boldsymbol{x} = \mathbf{C}\boldsymbol{x} + \boldsymbol{d}$, the vector $\mathbf{C}\boldsymbol{x}$ represents:

A. final demand

B. internal sector demand

C. the residual

D. an inverse matrix

MCQ answers: 1-A, 2-B, 3-B

Practice exercises

1. For $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 5 \\ -2 & 0 \end{bmatrix}$, find $\mathbf{A}+\mathbf{B}$ and $2\mathbf{A}$.
2. Calculate the product in the worked example on page 4 without looking at the solution.
3. Find $\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$.
4. Find the inverse of $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and solve $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$.
5. Decide whether $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is invertible.
6. For $\mathbf{C} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 70 \\ 50 \end{pmatrix}$, solve $(\mathbf{I} - \mathbf{C})\boldsymbol{x} = \boldsymbol{d}$.
7. Find all k for which $\begin{bmatrix} k & 2 \\ 3 & k \end{bmatrix}$ is singular.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem

A factory produces Products P and Q. Each unit of P requires 2 machine-hours and 1 labour-hour. Each unit of Q requires 1 machine-hour and 3 labour-hours. The available resources are 80 machine-hours and 90 labour-hours.

Define a production vector, formulate $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, use the determinant to test invertibility, solve using $\mathbf{A}^{-1}$, verify with Python, and interpret the result.

Brief answers to practice

Exercise	Brief answer
1	$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 4 \end{bmatrix}$, $2\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 6 & 8 \end{bmatrix}$
2	$\mathbf{AB} = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$
3	-2
4	$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$
5	No; the determinant is zero
6	$\boldsymbol{x} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$
7	$k = \pm\sqrt{6}$
Scenario challenge	$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $\det(\mathbf{A}) = 5$, $\boldsymbol{x} = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$ units

Lecture summary

- Matrix dimensions determine which operations are defined.
- Matrix multiplication combines row information with column information, and order matters.
- The determinant detects singularity and measures signed scaling.
- An inverse reverses a nonsingular matrix and gives $\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b}$ conceptually.
- Applied matrix models require mathematical verification and contextual interpretation.

Research and career connection

Matrices, determinants, and inverses support scientific computing, econometrics, optimization, cryptography, computer graphics, network models, control systems, and AI. In large applications, efficient numerical solvers replace hand calculation while preserving the same mathematical structure.

Key terms: order • entry • transpose • matrix product • determinant • singular • nonsingular • identity matrix • inverse • input-output model

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy documentation.