PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 03 • Vectors, Linear Combinations, Span, and Subspaces

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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Vectors represent quantities; span and subspaces describe all combinations that those quantities can generate.

1. Why Vectors Matter

From direction and displacement to data profiles and scientific measurements

A vector is an ordered mathematical object whose entries describe components of one quantity. A vector may represent physical displacement, colour intensity, product features, sensor measurements, or a user's preferences.

Connection with Lecture 2

Lecture 2 studied matrices as structured collections of numbers. A matrix can be viewed as a collection of column vectors, and matrix-vector multiplication forms linear combinations of those columns.

Learning outcomes

By the end of this lecture, students should be able to:

- represent quantities as vectors in $\mathbb{R}^n$;
- calculate vector sums, scalar multiples, norms, unit vectors, and distances;
- express a vector as a linear combination of other vectors;
- determine whether a target belongs to the span of given vectors;
- explain the geometric meaning of span in $\mathbb{R}^2$ and $\mathbb{R}^3$;
- test whether a subset of $\mathbb{R}^n$ is a subspace; and
- construct and analyse a recommendation profile using Python.

Vector notation

Vectors are written as bold lowercase symbols and parenthesized columns:

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n.$$

The order of the components matters. For example,

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Central idea
A vector describes one point or direction. A linear combination blends several vectors. The span is the complete collection of all blends that can be produced.

One object, many meanings

Context	Example vector	Meaning of components
Motion	$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$	Horizontal and vertical displacement
Digital colour	$\begin{pmatrix} 220 \\ 80 \\ 40 \end{pmatrix}$	Red, green, and blue intensity
Data science	$\begin{pmatrix} 18 \\ 72 \\ 165 \end{pmatrix}$	Age, mass, and height of one observation
Recommendation	$\begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$	Preference for three content categories

2. Vector Operations and Geometry

Let

$$\boldsymbol{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Addition and scalar multiplication

Operations are performed component by component:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} 3 + (-1) \\ -4 + 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \qquad 2\boldsymbol{v} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}.$$

Geometrically, vector addition follows the head-to-tail or parallelogram rule. Multiplication by a positive scalar changes length; a negative scalar also reverses direction.

Norm and unit vector

The Euclidean norm of $\boldsymbol{v} \in \mathbb{R}^n$ is

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

For the vector above,

$$\|\boldsymbol{v}\| = \sqrt{3^2 + (-4)^2} = 5.$$

The corresponding unit vector is

$$\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}, \qquad \|\widehat{\boldsymbol{v}}\| = 1.$$

Distance between vectors

The Euclidean distance between $\boldsymbol{v}$ and $\boldsymbol{w}$ is

$$d(\boldsymbol{v}, \boldsymbol{w}) = \|\boldsymbol{v} - \boldsymbol{w}\|.$$

Here,

$$\boldsymbol{v} - \boldsymbol{w} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \qquad d(\boldsymbol{v}, \boldsymbol{w}) = \sqrt{4^2 + (-6)^2} = 2\sqrt{13}.$$

Geometric laws

For scalars a, b and vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$:

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}, \qquad a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}, \qquad (a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}.$$

Interpretation
The norm measures the size of one vector. Distance measures how different two vector representations are. These ideas will support the recommendation application later in the lecture.

3. Linear Combinations

Building a target vector by scaling and adding known vectors

Given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$, a linear combination is any vector of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k, \qquad c_1, \ldots, c_k \in \mathbb{R}.$$

The scalars c_i are called weights or coefficients.

Worked example in $\mathbb{R}^2$

Determine whether

$$\boldsymbol{t} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

is a linear combination of

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We seek scalars a and b such that

$$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \boldsymbol{t}.$$

Equating components gives

$$\begin{aligned} a + 2b &= 5, \\ 2a + b &= 4. \end{aligned}$$

Solving yields $a = 1$ and $b = 2$. Therefore,

$$\boxed{\boldsymbol{t} = \boldsymbol{v}_1 + 2\boldsymbol{v}_2.}$$

Verification:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Matrix interpretation

Place the generating vectors into the columns of a matrix:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\boldsymbol{c}} = \underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{\boldsymbol{t}}.$$

Thus, checking a linear combination is equivalent to solving $\mathbf{M}\boldsymbol{c} = \boldsymbol{t}$.

Meaning of the coefficients
Coefficients tell us how much of each generating vector is needed. Negative coefficients are allowed mathematically; in an application, however, negative quantities may or may not be meaningful.

Special cases

- If the system has a solution, the target belongs to the span.
- If no solution exists, the target cannot be produced from the given vectors.
- If many coefficient choices work, the generating set contains redundancy.

4. Span and Its Geometric Meaning

The span of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is

$$\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = \{c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k : c_i \in \mathbb{R}\}.$$

Geometric possibilities

Setting	Generating situation	Span
$\mathbb{R}^2$	One nonzero vector	A line through the origin
$\mathbb{R}^2$	Two vectors in different directions	The entire plane $\mathbb{R}^2$
$\mathbb{R}^3$	One nonzero vector	A line through the origin
$\mathbb{R}^3$	Two nonparallel vectors	A plane through the origin
$\mathbb{R}^3$	Three independent directions	The entire space $\mathbb{R}^3$

Worked example: a plane in $\mathbb{R}^3$

Let

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

A general linear combination is

$$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \begin{pmatrix} a \\ b \\ a+b \end{pmatrix}.$$

Therefore,

$$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z = x + y \right\},$$

a plane through the origin.

For

$$\boldsymbol{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

we have $1 = 2 + (-1)$, so

$$\boldsymbol{u} = 2\boldsymbol{v}_1 - \boldsymbol{v}_2$$

and $\boldsymbol{u}$ belongs to the span.

For

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the condition $z = x + y$ would require $1 = 2$, which is false. Hence $\boldsymbol{w}$ does not belong to the span.

Span always contains the zero vector
Choose every coefficient equal to zero. This gives $\mathbf{0}$, so every span passes through the origin and is automatically a subspace.

5. Subspaces of $\mathbb{R}^n$

Sets of vectors that remain closed under the basic vector operations

A nonempty set $W \subseteq \mathbb{R}^n$ is a subspace if:

1. $\mathbf{0} \in W$;
2. whenever $\boldsymbol{u}, \boldsymbol{v} \in W$, then $\boldsymbol{u} + \boldsymbol{v} \in W$;
3. whenever $\boldsymbol{u} \in W$ and $c \in \mathbb{R}$, then $c\boldsymbol{u} \in W$.

Example: a subspace

Consider

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + 2y - z = 0 \right\}.$$

The zero vector satisfies the equation. If $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy it, adding their equations shows that $\boldsymbol{u} + \boldsymbol{v}$ also satisfies it. Multiplying the equation by any scalar c shows that $c\boldsymbol{u}$ also satisfies it. Therefore W is a subspace.

Solving for z gives $z = x + 2y$, so every vector in W has the form

$$\begin{pmatrix} x \\ y \\ x + 2y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Thus,

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Nonexample: a shifted line

Let

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + y = 1 \right\}.$$

The zero vector does not satisfy $0 + 0 = 1$, so $\mathbf{0} \notin S$. Therefore S is not a subspace. Geometrically, it is a line that does not pass through the origin.

> **Fast first check**
> Test the zero vector before checking closure. If $\mathbf{0}$ is absent, the set is immediately ruled out as a subspace. Passing the zero-vector test is necessary but not sufficient; closure must still be checked.

Important examples of subspaces

- the span of any collection of vectors;
- the solution set of a homogeneous system $\mathbf{M}\boldsymbol{x} = \mathbf{0}$;
- every line or plane through the origin; and
- $\{\mathbf{0}\}$ and $\mathbb{R}^n$ itself.

6. Application Case Study: A Recommendation Profile

Representing content and preferences as vectors in a feature space

A streaming platform represents each movie by three features:

$$\text{Action}, \qquad \text{Comedy}, \qquad \text{Science Fiction}.$$

Each movie is a vector in $\mathbb{R}^3$, and its components add to 1.

A user's viewing history

Suppose the user watched three movies with feature vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 0.8 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}.$$

More recent or important movies receive weights

$$\boldsymbol{w} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \end{pmatrix}.$$

Place the movie vectors in the columns of a matrix:

$$\mathbf{M} = \begin{bmatrix} 0.8 & 0.6 & 0.4 \\ 0.1 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}.$$

The user profile is the weighted linear combination

$$\boldsymbol{u} = \mathbf{M}\boldsymbol{w} = 0.5\boldsymbol{v}_1 + 0.3\boldsymbol{v}_2 + 0.2\boldsymbol{v}_3 = \boxed{\begin{pmatrix} 0.66 \\ 0.18 \\ 0.16 \end{pmatrix}}.$$

Thus $\boldsymbol{u} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Candidate movies

$$\boldsymbol{c}_A = \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix}, \quad \boldsymbol{c}_B = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}, \quad \boldsymbol{c}_C = \begin{pmatrix} 0.6 \\ 0.1 \\ 0.3 \end{pmatrix}.$$

Compare each candidate with the user profile using Euclidean distance:

$$d(\boldsymbol{u}, \boldsymbol{c}_A) \approx 0.0748, \qquad d(\boldsymbol{u}, \boldsymbol{c}_B) \approx 0.4490, \qquad d(\boldsymbol{u}, \boldsymbol{c}_C) \approx 0.1720.$$

Recommendation decision
Candidate A is closest to the user profile and is therefore recommended. This decision uses vector representation, a weighted linear combination, and geometric distance.

Important limitation
This model uses only three declared features. A real system should also consider data quality, diversity, changing preferences, privacy, and repeated recommendations that may narrow a user's exposure.

7. Python Laboratory

Construct, compare, and visualize vectors in the recommendation feature space

```
import numpy as np
import matplotlib.pyplot as plt

# Viewing-history vectors are stored as columns
M = np.array([
    [0.8, 0.6, 0.4],
    [0.1, 0.3, 0.2],
    [0.1, 0.1, 0.4]
], dtype=float)

weights = np.array([0.5, 0.3, 0.2], dtype=float)
profile = M @ weights

candidates = np.array([
    [0.7, 0.2, 0.1],
    [0.3, 0.3, 0.4],
    [0.6, 0.1, 0.3]
], dtype=float)

distances = np.linalg.norm(candidates - profile, axis=1)
best = np.argmin(distances)

print("User profile:", profile)
print("Distances:", np.round(distances, 4))
print("Recommended candidate:", chr(65 + best))

# 3D visualization of the feature vectors
fig = plt.figure(figsize=(7, 5))
ax = fig.add_subplot(111, projection="3d")
ax.scatter(*profile, s=130, label="User profile")
ax.scatter(candidates[:, 0], candidates[:, 1],
           candidates[:, 2], s=80, label="Candidates")
ax.set_xlabel("Action")
ax.set_ylabel("Comedy")
ax.set_zlabel("Science Fiction")
ax.legend()
plt.show()
```

Expected output

```
User profile: [0.66 0.18 0.16]
Distances: [0.0748 0.4490 0.1720]
Recommended candidate: A
```

Interpretation

- `M @ weights` forms the linear combination $\mathbf{M}\boldsymbol{w}$.
- `axis=1` computes one distance for each candidate row.
- `argmin` returns the position of the smallest distance.
- The plot shows the profile and candidates as points in $\mathbb{R}^3$.

Code and notation
NumPy uses square brackets for array storage. In mathematical writing, matrices use square brackets while vectors are shown as bold parenthesized columns.

8. Concept Check and Practice

Common misconceptions

Misconception	Why it fails	Correct practice
Vectors are only arrows	Vectors also represent data and features	Interpret components from context
Span means ordinary addition	Arbitrary scalar coefficients are allowed	Solve for the coefficients
Any line is a subspace	A subspace must contain $\mathbf{0}$	Check the origin and closure
The nearest item is always best	Distance depends on features and scaling	Interpret the model and its limitations

Three concept-check MCQs

1. The span of one nonzero vector in $\mathbb{R}^3$ is generally:

A. a point

B. a line through the origin

C. a plane

D. all of $\mathbb{R}^3$

2. Which set is not a subspace of $\mathbb{R}^2$?

A. $\{\mathbf{0}\}$

B. $\mathbb{R}^2$

C. $\{(x, y) : x + y = 0\}$

D. $\{(x, y) : x + y = 1\}$

3. If $\boldsymbol{t} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ has a solution for a, b, then:

A. $\boldsymbol{t}$ is outside the span

B. $\boldsymbol{t}$ belongs to the span

C. $\boldsymbol{t} = \mathbf{0}$

D. $a = b = 0$

MCQ answers: 1-B, 2-D, 3-B

Practice exercises

1. For $\boldsymbol{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find $\boldsymbol{v} + \boldsymbol{w}$, $2\boldsymbol{v} - \boldsymbol{w}$, and $\|\boldsymbol{v}\|$.
2. Find the unit vector in the direction of $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$.
3. Express $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
4. Describe span $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$ geometrically.
5. Decide whether $W = \{(x, y, z) : x - y + z = 0\}$ is a subspace of $\mathbb{R}^3$.
6. Explain why $S = \{(x, y) : x + y = 1\}$ is not a subspace.
7. Recalculate the recommendation profile for $\boldsymbol{w} = \begin{pmatrix} 0.2 & 0.5 & 0.3 \end{pmatrix}^{\mathsf{T}}$.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem

A water-treatment team uses two materials whose contaminant-removal profiles are

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Determine whether the target profile $\boldsymbol{t} = \begin{pmatrix} 5 & 4 & 1 \end{pmatrix}^{\mathsf{T}}$ belongs to their span. Find the required coefficients, verify using Python, and interpret whether the coefficients are practically meaningful.

Brief answers to practice

Exercise	Brief answer
1	$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $2\boldsymbol{v} - \boldsymbol{w} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$, $\lVert \boldsymbol{v} \rVert = \sqrt{5}$
2	$\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$
3	$3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
4	A line through the origin: span$\{(1,2)^{\mathsf{T}}\}$
5	Yes; it is the solution set of a homogeneous linear equation
6	No; $\mathbf{0}$ does not satisfy $x + y = 1$
7	$\boldsymbol{u} = \begin{pmatrix} 0.58 \\ 0.23 \\ 0.19 \end{pmatrix}$
Scenario challenge	$\boldsymbol{t} = 2\boldsymbol{v}_1 + \boldsymbol{v}_2$; use 2 units of Material 1 and 1 unit of Material 2

Lecture summary

- Vectors represent directions, measurements, and data profiles.
- Linear combinations build new vectors from weighted generating vectors.
- Span contains every vector obtainable from a given set.
- A subspace contains $\mathbf{0}$ and is closed under addition and scalar multiplication.
- Recommendation models require both mathematical verification and responsible interpretation.

Research and career connection

Vector spaces support computer graphics, data science, signal processing, robotics, optimization, network analysis, natural-language processing, and AI embeddings. Later lectures will study independence, basis, dimension, orthogonality, projections, and decompositions within these spaces.

Key terms: vector • component • norm • unit vector • distance • linear combination • coefficient • span • subspace • closure • feature space

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy documentation.