PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 04 • Linear Independence, Basis, Dimension, and Rank

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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Independence removes redundancy; a basis gives efficient coordinates; dimension and rank measure usable mathematical information.

1. From Span to Independence

When does every generating vector contribute genuinely new information?

Lecture 3 introduced span: all vectors obtainable through linear combinations. A spanning set may contain unnecessary vectors. Linear independence identifies whether any vector can already be built from the others.

Learning outcomes

By the end of this lecture, students should be able to:

- test a finite set of vectors for linear independence;
- construct and verify a basis for a vector space or subspace;
- interpret coordinates relative to a chosen basis;
- calculate dimension, matrix rank, and nullity;
- connect pivot positions with bases for column and row spaces; and
- use Python to identify redundant sensor channels in an application.

Formal definition

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are **linearly independent** if

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = c_2 = \cdots = c_k = 0.$$

They are **linearly dependent** if a nontrivial solution exists; at least one coefficient is not zero.

> **Central question**
> Does each vector introduce a new direction, or can one vector be recreated from the others? Independence means every vector contributes something essential.

Immediate observations

Situation	Conclusion	Reason
A set contains $\mathbf{0}$	Dependent	$1\mathbf{0} = \mathbf{0}$ is nontrivial
Two vectors are scalar multiples	Dependent	One is generated by the other
More than n vectors in $\mathbb{R}^n$	Dependent	There are not enough independent directions
Standard unit vectors in $\mathbb{R}^n$	Independent	Each controls a different coordinate

2. Testing Linear Independence

Convert the vector equation into a homogeneous linear system

Worked example: three vectors in $\mathbb{R}^3$

Let

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

To test independence, solve

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}.$$

Place the vectors in the columns of a matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad \mathbf{A}\boldsymbol{c} = \mathbf{0}, \qquad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Row reduction gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

There is a pivot in every column, so $c_1 = c_2 = c_3 = 0$. Therefore the vectors are linearly independent.

Dependent example

Now consider

$$\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{w}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Because

$$\boldsymbol{w}_1 + \boldsymbol{w}_2 - \boldsymbol{w}_3 = \mathbf{0},$$

the coefficient choice $(1, 1, -1)$ is nontrivial. Hence the set is dependent and $\boldsymbol{w}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ is redundant.

Equivalent matrix test
The columns of $\mathbf{A}$ are linearly independent exactly when $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ has only the zero solution. For a square matrix, this is also equivalent to $\det(\mathbf{A}) \neq 0$ and $\mathbf{A}^{-1}$ existing.

A practical testing workflow

1. Put the vectors into the columns of a matrix.
2. Row-reduce the matrix.
3. Identify pivot and free-variable columns.
4. A pivot in every column means independence; any free variable means dependence.

3. Basis and Coordinates

A basis is a complete set of directions with no redundancy

A set $\mathcal{B} = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_k\}$ is a **basis** for a vector space V if:

1. $\mathcal{B}$ spans V; and
2. $\mathcal{B}$ is linearly independent.

> **Two conditions, one efficient representation**
> Spanning guarantees that every vector in the space can be represented. Independence guarantees that the representation is unique.

Standard bases

The standard basis of $\mathbb{R}^3$ is

$$\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Every $\boldsymbol{v} = (v_1, v_2, v_3)^{\mathsf{T}}$ has the unique representation

$$\boldsymbol{v} = v_1\boldsymbol{b}_1 + v_2\boldsymbol{b}_2 + v_3\boldsymbol{b}_3.$$

For the polynomial space P_2, a standard basis is $\{1, x, x^2\}$ because every polynomial $p(x) = a + bx + cx^2$ has unique coefficients.

Basis for a subspace

Consider the plane

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}.$$

Since $z = x + y$,

$$\begin{pmatrix} x \\ y \\ x + y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus a basis is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The two basis vectors are independent and span every vector satisfying $x + y - z = 0$.

Coordinates relative to a basis

For $\boldsymbol{u} = (3, -2, 1)^{\mathsf{T}} \in W$,

$$\boldsymbol{u} = 3\boldsymbol{b}_1 - 2\boldsymbol{b}_2, \qquad [\boldsymbol{u}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Coordinates depend on the chosen basis, while the vector itself does not change.

4. Dimension

Counting the independent directions needed to describe a space

The **dimension** of a finite-dimensional vector space V is the number of vectors in any basis for V:

$$\dim(V) = \text{number of basis vectors.}$$

Examples

Space	One possible basis	Dimension
$\mathbb{R}^2$	$\{(1,0)^\top, (0,1)^\top\}$	2
$\mathbb{R}^3$	Standard unit vectors	3
A line through the origin	One nonzero direction vector	1
A plane through the origin in $\mathbb{R}^3$	Two independent vectors in the plane	2
P_2	$\{1, x, x^2\}$	3
The zero space $\{\mathbf{0}\}$	Empty basis	0

Why all bases have the same size

Different bases may use different vectors, but they must contain the same number of vectors. Therefore dimension belongs to the space, not to a particular basis.

For example, both

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

are bases for $\mathbb{R}^2$, and both contain two vectors.

Basis reduction and extension

- From a spanning set, remove dependent vectors until a basis remains.
- From an independent set, add suitable vectors until it becomes a basis for the full space.

Dimension as degrees of freedom
Dimension measures how many independent parameters are required. A point has zero degrees of freedom, a line has one, a plane has two, and $\mathbb{R}^n$ has n.

Important consequence in $\mathbb{R}^n$

Any set of more than n vectors is dependent. Any set of exactly n independent vectors automatically forms a basis for $\mathbb{R}^n$.

5. Matrix Rank and the Rank-Nullity Theorem

Rank measures independent output directions; nullity measures invisible input directions

For an $m \times n$ matrix $\mathbf{A}$:

- the **column space** $\text{Col}(\mathbf{A})$ is the span of the columns;
- the **row space** $\text{Row}(\mathbf{A})$ is the span of the rows;
- the **null space** $\text{Nul}(\mathbf{A})$ contains all $\boldsymbol{x}$ satisfying $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

The rank is

$$\text{rank}(\mathbf{A}) = \dim(\text{Col}(\mathbf{A})) = \dim(\text{Row}(\mathbf{A})).$$

Worked example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \sim \mathbf{R} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivot columns, so

$$\text{rank}(\mathbf{A}) = 2.$$

The pivot positions occur in columns 1 and 2. A basis for the column space must use the corresponding columns of the original matrix:

$$\mathcal{B}_{\text{Col}(\mathbf{A})} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Null space and nullity

The reduced equations are

$$x_1 + x_3 + 2x_4 = 0, \qquad x_2 + x_3 + x_4 = 0.$$

With $x_3 = s$ and $x_4 = t$,

$$\boldsymbol{x} = s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore $\text{nullity}(\mathbf{A}) = 2$. Since $\mathbf{A}$ has four columns,

$$\boxed{\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4.}$$

Interpretation
Rank counts independent column directions. Nullity counts independent input combinations that the matrix maps to zero. Their sum equals the number of input variables.

6. Application Case Study: Redundant Sensor Channels

Selecting an informative basis for an environmental monitoring system

An air-quality monitoring station has four sensor channels. Engineers test each channel under three calibration conditions. The response profile of each channel is a vector in $\mathbb{R}^3$:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}1\\1\\2\end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix}2\\1\\3\end{pmatrix}.$$

Problem

Determine how many independent sensor-response patterns exist. Identify a basis, explain the redundant channels, and state why rank alone does not decide which physical sensor should be removed.

Mathematical model

Store the response vectors as columns:

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 1 & 2\\0 & 1 & 1 & 1\\1 & 1 & 2 & 3\end{bmatrix}.$$

Row reduction produces two pivots, so rank$(\mathbf{A}) = 2$. The first two original columns form a basis:

$$\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

The remaining profiles satisfy

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2, \qquad \boldsymbol{v}_4 = 2\boldsymbol{v}_1 + \boldsymbol{v}_2.$$

Thus four recorded channels contain only two independent response patterns.

Operational interpretation

Finding	Mathematical meaning	Engineering meaning
Rank $= 2$	Two independent column directions	Two response patterns carry the linear information
$\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2$	Column 3 is dependent	Channel 3 is reproducible in this calibration model
$\boldsymbol{v}_4 = 2\boldsymbol{v}_1 + \boldsymbol{v}_2$	Column 4 is dependent	Channel 4 adds no new linear direction

Decision
Retaining channels 1 and 2 preserves the column space of the calibration matrix. This gives a mathematically compact representation of the observed response patterns.

Responsible limitation
Linear dependence is not the only engineering criterion. Cost, noise, reliability, physical location, sensitivity, and safety requirements must also be considered before removing a real sensor.

7. Python Laboratory

Verify rank, obtain pivot columns, expose dependencies, and visualize the data

```
import numpy as np
import matplotlib.pyplot as plt

def rref(matrix, tol=1e-10):
    """Return reduced row-echelon form and pivot columns."""
    R = matrix.astype(float).copy()
    pivots, row = [], 0
    for col in range(R.shape[1]):
        choices = np.flatnonzero(np.abs(R[row:, col]) > tol)
        if choices.size == 0:
            continue
        pivot = row + choices[0]
        R[[row, pivot]] = R[[pivot, row]]
        R[row] = R[row] / R[row, col]
        for other in range(R.shape[0]):
            if other != row:
                R[other] -= R[other, col] * R[row]
        pivots.append(col)
        row += 1
        if row == R.shape[0]:
            break
    R[np.abs(R) < tol] = 0
    return R, tuple(pivots)

# Columns are sensor-response vectors
A = np.array([
    [1, 0, 1, 2],
    [0, 1, 1, 1],
    [1, 1, 2, 3]
], dtype=float)

rank = np.linalg.matrix_rank(A)
R, pivots = rref(A)
basis = A[:, list(pivots)]

print("Rank:", rank)
print("Pivot columns (0-based):", pivots)
print("Basis columns:\n", basis)
print("RREF:\n", R)

# Verify the two dependent channels
print("v3 = v1 + v2:", np.allclose(A[:, 2], A[:, 0] + A[:, 1]))
print("v4 = 2v1 + v2:", np.allclose(A[:, 3], 2*A[:, 0] + A[:, 1]))

# Heat map of channel responses across calibration conditions
fig, ax = plt.subplots(figsize=(7, 3.5))
image = ax.imshow(A, cmap="Blues", aspect="auto")
ax.set_xticks(range(4), ["Channel 1", "Channel 2", "Channel 3", "Channel 4"])
ax.set_yticks(range(3), ["Condition 1", "Condition 2", "Condition 3"])
for i in range(A.shape[0]):
    for j in range(A.shape[1]):
        ax.text(j, i, f"{A[i, j]:.0f}", ha="center", va="center")
fig.colorbar(image, ax=ax, label="Response")
plt.tight_layout()
plt.show()
```

Selected output

```
Rank: 2
Pivot columns (0-based): (0, 1)
Basis columns:
 [[1. 0.]
  [0. 1.]
  [1. 1.]]
v3 = v1 + v2: True
v4 = 2v1 + v2: True
```

Interpretation
The pivot indices identify original columns 1 and 2 as one basis. The direct comparisons expose exact dependencies among the four channels. The NumPy and Matplotlib code runs directly in Google Colab; the engineer must still interpret whether the model is suitable for action.

8. Concept Check and Practice

Common misconceptions

Misconception	Why it fails	Correct practice
A spanning set is always a basis	It may contain redundant vectors	Check both span and independence
Row operations preserve column vectors	Row reduction changes the columns	Use pivot positions but select original columns
Rank is the number of rows	Some rows may be dependent	Count pivots after row reduction
Any two vectors form a basis of $\mathbb{R}^2$	Parallel vectors are dependent	Verify independence

Three concept-check MCQs

1. A set containing the zero vector is:

A. always independent

B. always dependent

C. a basis

D. independent only in $\mathbb{R}^2$

2. If a 3×5 matrix has rank 3, its nullity is:

A. 0

B. 2

C. 3

D. 5

3. Which pair is a basis for $\mathbb{R}^2$?

A. $\{(1,2)^\top, (2,4)^\top\}$

B. $\{(0,0)^\top, (1,1)^\top\}$

C. $\{(1,1)^\top, (1,-1)^\top\}$

D. $\{(1,0)^\top\}$

MCQ answers: 1-B, 2-B, 3-C

Practice exercises

1. Determine whether $\{(1,2)^\top, (3,6)^\top\}$ is linearly independent.
2. Test whether the vectors $(1,0,1)^\top$, $(0,1,1)^\top$, and $(1,1,2)^\top$ are independent.
3. Find a basis and dimension for $W = \{(x,y,z)^\top : x - y + z = 0\}$.
4. Show that $\{1 + x,\ x + x^2,\ 1 - x^2\}$ is dependent in P_2.
5. Find the rank and nullity of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.
6. Give a basis for the column space of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.
7. Modify the sensor matrix by replacing channel 4 with $(0,0,1)^\top$. Use Python to calculate the new rank and interpret the result.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem

A data-compression team represents four signals by

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the dimension of their span, select a basis from the original signals, and explain how the result supports compression.

Brief answers to practice

Exercise	Brief answer
1	Dependent because $(3,6)^\top = 3(1,2)^\top$
2	Dependent because the third vector is the sum of the first two
3	One basis is $\{(1,1,0)^\top, (-1,0,1)^\top\}$; $\dim(W) = 2$
4	$(1+x) - (x+x^2) - (1-x^2) = 0$
5	Rank $= 1$ and nullity $= 2$
6	Use original columns 1 and 3: $\{(1,0,1)^\top, (0,1,1)^\top\}$
7	New rank $= 3$; the replacement adds a third independent response direction
Scenario	Dimension $= 3$; one basis is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$. The four signals can be represented using three independent directions.

Lecture summary

- Independence means no vector is reproducible from the others.
- A basis is an independent spanning set and gives unique coordinates.
- Dimension counts the basis vectors, or independent degrees of freedom.
- Rank is the dimension of the column space and equals the row-space dimension.
- Rank-nullity divides input variables into visible and zero-output directions.
- Applications require mathematical verification and domain-aware interpretation.

Research and career connection

Basis selection and rank appear in data compression, feature engineering, numerical analysis, network science, control systems, signal processing, scientific computing, and machine learning. Later lectures will use these concepts in transformations, least squares, eigenvectors, SVD, and dimensionality reduction.

Key terms: linear independence • dependence • basis • coordinates • dimension • column space • row space • null space • rank • nullity • pivot

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy matrix-rank documentation; SymPy matrix documentation.