PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 05 • Linear Transformations and Matrix Representations

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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A linear transformation is a rule that preserves vector structure; its matrix records what the rule does to basis directions.

1. Understanding Linear Transformations

A structure-preserving map between vector spaces

A transformation $T : V \to W$ assigns each vector in the domain V exactly one vector in the codomain W. It is **linear** when, for all vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ and scalars c,

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}), \qquad T(c\boldsymbol{v}) = cT(\boldsymbol{v}).$$

Equivalently,

$$T(a\boldsymbol{u} + b\boldsymbol{v}) = aT(\boldsymbol{u}) + bT(\boldsymbol{v}).$$

Learning outcomes

By the end of this lecture, students should be able to:

- verify whether a given rule is a linear transformation;
- identify domain, codomain, range, and kernel;
- construct the standard matrix of a linear transformation;
- interpret scaling, rotation, reflection, shear, and projection matrices;
- calculate compositions and inverse transformations;
- connect rank and nullity with one-to-one and onto mappings; and
- solve and visualize a robotic coordinate-transformation problem in Python.

Essential consequence

Every linear transformation satisfies

$$T(\mathbf{0}) = \mathbf{0}.$$

Indeed, $T(\mathbf{0}) = T(0\boldsymbol{v}) = 0T(\boldsymbol{v}) = \mathbf{0}$. Therefore a rule that moves the origin cannot be linear.

Quick classification

Rule	Linear?	Reason
$T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$	Yes	Matrix multiplication preserves sums and scalar multiples
$F(x, y) = (x + 1, y)$	No	$F(0, 0) \neq (0, 0)$
$G(x, y) = (x^2, y)$	No	Squaring does not preserve scalar multiplication
$P(x, y) = (x, 0)$	Yes	Projection onto the x-axis preserves linear combinations

Central idea
Linearity means that transforming a linear combination gives the same result as transforming each vector first and then forming the same linear combination.

2. Matrix Representation

The columns reveal the images of the standard basis vectors

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x},$$

where $\mathbf{A}$ is an $m \times n$ matrix. Its jth column is $T(\boldsymbol{e}_j)$, the image of the jth standard basis vector.

Constructing the standard matrix

Suppose

$$T(x, y) = \begin{pmatrix} 2x - y \\ x + 3y \\ 2y \end{pmatrix}.$$

Using

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we obtain

$$T(\boldsymbol{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad T(\boldsymbol{e}_2) = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}.$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}, \qquad T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}.$$

For $\boldsymbol{x} = (2, -1)^\mathsf{T}$,

$$T(\boldsymbol{x}) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}.$$

Why the columns are sufficient

Every vector $\boldsymbol{x} = (x_1, \ldots, x_n)^\mathsf{T}$ is a linear combination of the standard basis vectors:

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n.$$

By linearity,

$$T(\boldsymbol{x}) = x_1 T(\boldsymbol{e}_1) + \cdots + x_n T(\boldsymbol{e}_n).$$

Thus the action on a basis determines the action on the entire space.

Dimension check
If $T : \mathbb{R}^n \to \mathbb{R}^m$, then its standard matrix has m rows and n columns. The input vector has n components and the output has m components.

Matrix representation relative to other bases

The transformation is independent of coordinates, but its matrix depends on the selected bases. Changing bases changes the coordinate description, not the underlying mapping.

3. Geometric Transformations in $\mathbb{R}^2$

Matrices reshape directions, lengths, angles, and area

Transformation	Matrix	Geometric effect
Scaling	$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$	Stretches or compresses along coordinate axes
Rotation by θ	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Rotates counterclockwise about the origin
Reflection in x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reverses the vertical component
Horizontal shear	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	Shifts horizontal position according to height
Projection onto x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Removes the vertical component

Worked example: rotation

A counterclockwise rotation by 90° is represented by

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $\boldsymbol{v} = (3, 1)^\mathsf{T}$,

$$\mathbf{R}\boldsymbol{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

The length is preserved:

$$\|\boldsymbol{v}\| = \sqrt{10} = \|\mathbf{R}\boldsymbol{v}\|.$$

Area and orientation

For a 2×2 transformation matrix $\mathbf{A}$, the factor $|\det(\mathbf{A})|$ describes how area changes.

- $|\det(\mathbf{A})| > 1$: area expands.
- $0 < |\det(\mathbf{A})| < 1$: area contracts.
- $\det(\mathbf{A}) = 0$: the plane collapses into a line or point.
- $\det(\mathbf{A}) < 0$: orientation is reversed.

Computer graphics connection
Graphics systems use transformation matrices to reposition models, rotate objects, scale images, reflect shapes, and project three-dimensional scenes onto display coordinates.

4. Kernel, Range, and Mapping Properties

What information disappears, and which outputs can be reached?

For a linear transformation $T : V \to W$:

$$\ker(T) = \{\boldsymbol{x} \in V : T(\boldsymbol{x}) = \mathbf{0}\}, \qquad \operatorname{range}(T) = \{T(\boldsymbol{x}) : \boldsymbol{x} \in V\}.$$

If $T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$, then

$$\ker(T) = \operatorname{Nul}(\mathbf{A}), \qquad \operatorname{range}(T) = \operatorname{Col}(\mathbf{A}).$$

Worked example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ have matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The matrix has two pivots, so

$$\operatorname{rank}(\mathbf{A}) = 2, \qquad \operatorname{range}(T) = \mathbb{R}^2.$$

To find the kernel, solve

$$x_1 + 2x_2 = 0, \qquad x_2 + x_3 = 0.$$

Let $x_2 = s$. Then

$$\boldsymbol{x} = s \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \qquad \ker(T) = \operatorname{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Thus $\operatorname{nullity}(\mathbf{A}) = 1$, and

$$\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = 2 + 1 = 3.$$

One-to-one and onto

Property	Linear-algebra test	Meaning
One-to-one	$\ker(T) = \{\mathbf{0}\}$; pivot in every input column	Different inputs cannot produce the same output
Onto	$\operatorname{range}(T) = W$; pivot in every output row	Every vector in the codomain can be produced
Invertible	One-to-one and onto	Every output has exactly one input

The example is onto $\mathbb{R}^2$ but not one-to-one because its kernel contains nonzero vectors.

Information interpretation
Kernel directions are lost by the transformation. Range directions are the outputs that survive. Rank counts reachable independent directions; nullity counts lost independent directions.

5. Composition and Inverse Transformations

Multiple transformations combine through matrix multiplication

If $T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$ and $S(\boldsymbol{y}) = \mathbf{B}\boldsymbol{y}$, then

$$(S \circ T)(\boldsymbol{x}) = S(T(\boldsymbol{x})) = \mathbf{BA}\boldsymbol{x}.$$

The rightmost matrix acts first.

Worked composition: rotate, then scale

Let

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Rotating first and scaling second gives

$$\mathbf{SR} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$

For $\boldsymbol{v} = (1,2)^{\mathsf{T}}$,

$$(\mathbf{SR})\boldsymbol{v} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

Reversing the order gives

$$\mathbf{RS} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \qquad (\mathbf{RS})\boldsymbol{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Therefore composition order generally matters.

Inverse transformation

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible when its matrix $\mathbf{A}$ is invertible. Then

$$T^{-1}(\boldsymbol{y}) = \mathbf{A}^{-1}\boldsymbol{y}.$$

For

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

If $\boldsymbol{y} = (5,3)^{\mathsf{T}}$, then

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Verification gives $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$.

Invertibility theorem for square matrices
For $\mathbf{A} \in \mathbb{R}^{n \times n}$, the following are equivalent: $\mathbf{A}$ is invertible; $\det(\mathbf{A}) \neq 0$; $\operatorname{rank}(\mathbf{A}) = n$; the columns form a basis of $\mathbb{R}^n$; and $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has only the zero solution.

Stable computation

In numerical work, solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$ directly rather than explicitly computing $\mathbf{A}^{-1}$. This is usually faster and more stable.

6. Application Case Study: Robot Coordinate Mapping

Transform local sensor coordinates into a global map

A mobile robot detects three landmarks in its local coordinate system. Store the landmark vectors as columns:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The robot is rotated 90° counterclockwise relative to the global map, and its global position is

$$\boldsymbol{t} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Step 1: rotate the local coordinates

Use

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\mathbf{RP} = \begin{bmatrix} -1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Step 2: translate into the global map

Add $\boldsymbol{t}$ to every rotated landmark:

$$\mathbf{Q} = \mathbf{RP} + \boldsymbol{t}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 5 & 3 & 2 \end{bmatrix}.$$

Thus the global landmark vectors are $(3,5)^\top$, $(2,3)^\top$, and $(3,2)^\top$.

Linear versus affine

Rotation is linear because it preserves the origin. Translation is not linear because it moves $\mathbf{0}$ to $\boldsymbol{t}$. Together they form an **affine transformation**:

$$\boldsymbol{q} = \mathbf{R}\boldsymbol{p} + \boldsymbol{t}.$$

Using homogeneous coordinates, both operations can be represented by one matrix:

$$\mathbf{H} = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{pmatrix} q_x \\ q_y \\ 1 \end{pmatrix} = \mathbf{H}\begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}.$$

Application decision
The robot should use the transformed landmarks **Q** when updating the global map. The original matrix **P** remains meaningful only in the robot's local frame.

Responsible limitation
Real mapping also includes sensor noise, uncertain robot pose, calibration error, and changing environments. A transformation model is necessary, but uncertainty must also be estimated.

7. Python Laboratory

Compute, verify, reverse, and visualize the robot transformation

```
import numpy as np
import matplotlib.pyplot as plt

# Local landmark vectors are columns
P = np.array([
    [2.0, 0.0, -1.0],
    [1.0, 2.0,  1.0]
])

theta = np.deg2rad(90)
R = np.array([
    [np.cos(theta), -np.sin(theta)],
    [np.sin(theta),  np.cos(theta)]
])
t = np.array([[4.0], [3.0]])

# Affine transformation: rotate, then translate
Q = R @ P + t

# Equivalent homogeneous-coordinate calculation
H = np.block([
    [R, t],
    [np.zeros((1, 2)), np.ones((1, 1))]
])
P_h = np.vstack([P, np.ones((1, P.shape[1]))])
Q_h = H @ P_h

# Reverse the mapping; R.T is the inverse of a rotation
P_recovered = R.T @ (Q - t)

print("Global landmarks:\n", np.round(Q, 6))
print("Homogeneous result agrees:", np.allclose(Q, Q_h[:2]))
print("Recovery agrees:", np.allclose(P, P_recovered))
print("det(R):", round(np.linalg.det(R), 6))

# Visual comparison of the two coordinate frames
fig, axes = plt.subplots(1, 2, figsize=(10, 4))
for ax, points, origin, title in [
    (axes[0], P, np.zeros((2, 1)), "Robot-local frame"),
    (axes[1], Q, t, "Global map frame")
]:
    ax.scatter(points[0], points[1], s=70, color="#1665D8")
    ax.scatter(origin[0], origin[1], marker="x", s=90, color="#E5A823")
    for j in range(points.shape[1]):
        ax.text(points[0, j] + 0.08, points[1, j] + 0.08, f"L{j+1}")
    ax.axhline(0, color="grey", lw=0.7)
    ax.axvline(0, color="grey", lw=0.7)
    ax.set_aspect("equal")
    ax.grid(alpha=0.25)
    ax.set_title(title)
plt.tight_layout()
plt.show()
```

Selected output

```
Global landmarks:
 [[3. 2. 3.]
  [5. 3. 2.]]
Homogeneous result agrees: True
Recovery agrees: True
det(R): 1.0
```

Interpretation
The determinant confirms that the rotation preserves area and orientation. Homogeneous coordinates reproduce the affine mapping, and the inverse calculation recovers the original local landmarks.

8. Concept Check and Practice

Common misconceptions

Misconception	Why it fails	Correct practice
Every transformation is linear	Translation and nonlinear rules may fail linearity	Test both linearity conditions
Matrix order does not matter	Matrix multiplication is not commutative	Apply the rightmost matrix first
Codomain and range are identical	Some codomain vectors may be unreachable	Compute the column space
An inverse always exists	Rank may be deficient	Check full rank or nonzero determinant

Three concept-check MCQs

1. Which rule is linear?

A. $T(x,y) = (x+2, y)$

B. $T(x,y) = (2x-y, x+y)$

C. $T(x,y) = (x^2, y)$

D. $T(x,y) = (|x|, y)$

2. The kernel of a one-to-one linear transformation is:

A. the domain

B. the codomain

C. $\{\mathbf{0}\}$

D. the range

3. If T has matrix $\mathbf{A}$ and S has matrix $\mathbf{B}$, then $S \circ T$ has matrix:

A. $\mathbf{A} + \mathbf{B}$

B. $\mathbf{AB}$

C. $\mathbf{BA}$

D. $\mathbf{A}^{-1}\mathbf{B}$

MCQ answers: 1-B, 2-C, 3-C

Practice exercises

1. Verify that $T(x,y) = (3x-y, 2y)$ is linear and find its matrix.
2. Show that $F(x,y) = (x, y+4)$ is not linear.
3. Find the image of $(2,-1)^{\mathsf{T}}$ under reflection in the x-axis.
4. Find the matrix for a 180° rotation and apply it to $(3,-2)^{\mathsf{T}}$.
5. For $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a basis for the kernel and range.
6. Compute $\mathbf{BA}$ for $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$; interpret the order.
7. Modify the robot problem to a 180° rotation with translation $(2,1)^{\mathsf{T}}$. Use Python to find the global landmarks.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem

A graphics program first applies the horizontal shear

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and then scales horizontal coordinates by 2 using

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the composite matrix, transform the vertex $\boldsymbol{v} = (1,2)^\mathsf{T}$, and explain why reversing the order changes the result.

Brief answers to practice

Exercise	Brief answer
1	$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$; matrix multiplication is linear
2	$F(0,0) = (0,4)^\mathsf{T} \neq \mathbf{0}$
3	$(2,1)^\mathsf{T}$
4	$\mathbf{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; image $= (-3,2)^\mathsf{T}$
5	$\ker(T) = \operatorname{span}\{(-1,1)^\mathsf{T}\}$; $\operatorname{range}(T) = \operatorname{span}\{(1,2)^\mathsf{T}\}$
6	$\mathbf{BA} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$; **A** acts first, then **B**
7	$\mathbf{Q} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & -1 & 0 \end{bmatrix}$
Scenario	$\mathbf{BA} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$ and $(\mathbf{BA})\boldsymbol{v} = (10,2)^\mathsf{T}$. Reversing the order gives $\mathbf{AB} \neq \mathbf{BA}$.

Lecture summary

- Linear transformations preserve vector addition and scalar multiplication.
- A matrix is constructed from the images of basis vectors.
- Kernel records lost directions; range records reachable outputs.
- Rank and nullity determine one-to-one and onto properties.
- Composition corresponds to matrix multiplication, with the rightmost matrix acting first.
- Translation is affine, but homogeneous coordinates combine it with linear operations.

Research and career connection

Transformations support computer graphics, robotics, computer vision, geographic information systems, signal processing, control theory, neural-network layers, numerical modelling, and scientific simulation.

Key terms: transformation • linearity • domain • codomain • range • kernel • standard matrix • rotation • reflection • shear • projection • composition • inverse • affine map

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy matrix-multiplication documentation; Matplotlib scatter documentation.