PMAI LAB

Linear Algebra: Theory, Computation, and Modern Applications

Lecture 08 • Eigenvalues, Eigenvectors, and Diagonalization

Course lens
Understand → Model → Compute → Apply → Code → Interpret

Audience
Undergraduate BS Mathematics students

Instructor
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Eigenvectors reveal directions preserved by a transformation; eigenvalues measure the corresponding change.

1. Why Eigenvalues and Eigenvectors?

Identify the directions in which a linear transformation acts most simply

A general vector may change both magnitude and direction under a square matrix **A**. An eigenvector is special: the transformation changes only its scale, and possibly reverses its direction.

$$\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}, \qquad \boldsymbol{v} \neq \mathbf{0}.$$

The nonzero vector $\boldsymbol{v}$ is an **eigenvector**, and the scalar λ is its **eigenvalue**.

Geometric meaning of λ

- $\lambda > 1$: stretch along the eigenvector direction.
- $0 < \lambda < 1$: contraction without reversal.
- $\lambda < 0$: scaling with direction reversal.
- $\lambda = 0$: collapse into the zero vector; therefore **A** is singular.
- $|\lambda| = 1$: magnitude is preserved along that direction.

Learning outcomes

By the end of this lecture, students should be able to:

- explain eigenvalues and eigenvectors geometrically and algebraically;
- form and solve the characteristic equation;
- compute eigenspaces from $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}$;
- determine whether a matrix is diagonalizable;
- use diagonalization to calculate matrix powers;
- interpret spectral radius and stability in a discrete system; and
- solve a transition-model application with NumPy.

Central idea
Eigenpairs replace a complicated transformation by independent modes. Each mode evolves by multiplication with one scalar, making repeated transformations and long-term behaviour easier to understand.

Where they appear

Population models, Markov chains, vibration modes, differential equations, quantum mechanics, network ranking, covariance analysis, recommendation systems, image processing, stability analysis, PCA, and machine learning.

2. Characteristic Equation and Eigenspaces

Convert the eigenvector equation into a determinant condition

Starting from $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$,

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}.$$

A nonzero solution exists only when $\mathbf{A} - \lambda\mathbf{I}$ is singular. Therefore,

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0.}$$

This is the **characteristic equation**; its polynomial is the characteristic polynomial.

For each eigenvalue λ, the corresponding eigenspace is

$$E_\lambda = \mathrm{Null}(\mathbf{A} - \lambda\mathbf{I}).$$

Every nonzero vector in E_λ is an eigenvector associated with λ.

Fast cases

If $\mathbf{A}$ is diagonal or triangular, its eigenvalues are its diagonal entries. For example,

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -2 & 5 \\ 0 & 0 & 6 \end{bmatrix} \implies \lambda = 3, -2, 6.$$

Useful spectral identities

For an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, counted with algebraic multiplicity,

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i, \qquad \det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i.$$

These provide quick checks after calculation.

Computation workflow
Step 1: Solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. **Step 2:** For each λ, solve $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}$. **Step 3:** Verify $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.

Multiplicity

Algebraic multiplicity is the number of times an eigenvalue occurs as a root. **Geometric multiplicity** is $\dim E_\lambda$. Always,

$$1 \le \dim E_\lambda \le \text{algebraic multiplicity}.$$

3. Worked Computation

Find both eigenvalues and their eigenvector directions step by step

Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} \\ &= (4-\lambda)(3-\lambda) - 2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 5)(\lambda - 2) = 0. \end{aligned}$$

Hence $\lambda_1 = 5$ and $\lambda_2 = 2$. The checks are

$$\lambda_1 + \lambda_2 = 7 = \operatorname{tr}(\mathbf{A}), \qquad \lambda_1 \lambda_2 = 10 = \det(\mathbf{A}).$$

Eigenvector for $\lambda_1 = 5$

$$(\mathbf{A} - 5\mathbf{I})\boldsymbol{v} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $-x + y = 0$ gives $y = x$. Choose

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 2$

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{v} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $2x + y = 0$ gives $y = -2x$. Choose

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Verification

$\mathbf{A}\boldsymbol{v}_1 = (5,5)^\top = 5\boldsymbol{v}_1$ and $\mathbf{A}\boldsymbol{v}_2 = (2,-4)^\top = 2\boldsymbol{v}_2$. Eigenvectors may be multiplied by any nonzero scalar without changing their direction.

4. Diagonalization and Matrix Powers

Change to an eigenvector basis so repeated transformations become simple

A matrix $\mathbf{A}$ is diagonalizable if it has n linearly independent eigenvectors. Put these eigenvectors into the columns of $\mathbf{P}$ and their eigenvalues into $\mathbf{D}$ in the same order:

$$\mathbf{P} = \begin{bmatrix} | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 \\ | & | \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Then

$$\mathbf{AP} = \mathbf{PD}, \qquad \boxed{\mathbf{A} = \mathbf{PDP}^{-1}.}$$

For the worked matrix,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}.$$

Why diagonalization is useful

Because adjacent factors cancel,

$$\mathbf{A}^k = (\mathbf{PDP}^{-1})^k = \mathbf{PD}^k\mathbf{P}^{-1}, \qquad \mathbf{D}^k = \begin{bmatrix} 5^k & 0 \\ 0 & 2^k \end{bmatrix}.$$

For example,

$$\mathbf{A}^4 = \mathbf{P} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \mathbf{P}^{-1} = \begin{bmatrix} 422 & 203 \\ 406 & 219 \end{bmatrix}.$$

Mode interpretation

If $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$, then

$$\mathbf{A}^k\boldsymbol{x}_0 = c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2.$$

Each eigenvector component evolves independently. When $|\lambda_1| > |\lambda_2|$ and $c_1 \neq 0$, the first mode eventually dominates the direction.

Diagonalizability test
Distinct eigenvalues guarantee independent eigenvectors. Repeated eigenvalues require care: the total number of independent eigenvectors must still equal the matrix size.

5. Spectral Structure and Stability

Use eigenvalues to classify transformations and predict long-term behaviour

Symmetric matrices

If $\mathbf{A}^\mathsf{T} = \mathbf{A}$, all eigenvalues are real, and eigenvectors belonging to distinct eigenvalues are orthogonal. A real symmetric matrix has an orthogonal diagonalization:

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T}, \qquad \mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}.$$

For

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

the eigenvalues are 3 and 1, with orthonormal eigenvectors

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Spectral radius and discrete stability

The spectral radius is

$$\rho(\mathbf{A}) = \max_i |\lambda_i|.$$

For the discrete system $\boldsymbol{x}_{k+1} = \mathbf{A}\boldsymbol{x}_k$:

- if all $|\lambda_i| < 1$, every mode decays toward zero;
- if some $|\lambda_i| > 1$, a corresponding component grows;
- if $|\lambda_i| = 1$, persistent or oscillatory behaviour may occur.

Two important exceptions

Defective matrix:

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has a repeated eigenvalue 3 but only one independent eigenvector, so it is not diagonalizable.

Complex eigenvalues:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

represents a 90° rotation and has eigenvalues i and $-i$. It has no real eigenvector direction.

> **Practical interpretation**
> The largest-magnitude eigenvalues usually govern long-term growth, decay, mixing, ranking, or variance. However, scaling, units, model assumptions, and numerical conditioning still matter.

6. Application Case Study: Two-Region Population Movement

Use a dominant eigenvector to find the long-run population distribution

Two regions, North and South, exchange residents each year. Of the North population, 80% remains and 20% moves South. Of the South population, 70% remains and 30% moves North. With a column population vector,

$$\boldsymbol{x}_{k+1} = \mathbf{M}\boldsymbol{x}_k, \qquad \mathbf{M} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}.$$

Each column sums to one, so total population proportion is preserved.

Eigenpairs and steady distribution

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5).$$

For $\lambda_1 = 1$,

$$(\mathbf{M} - \mathbf{I})\boldsymbol{v} = \mathbf{0} \quad \Longrightarrow \quad -0.2x + 0.3y = 0.$$

Thus $\boldsymbol{v}_1 = (3, 2)^{\mathsf{T}}$. Normalizing its entries to sum to one gives the steady distribution

$$\boxed{\boldsymbol{s} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}.}$$

For $\lambda_2 = 0.5$, choose $\boldsymbol{v}_2 = (1, -1)^{\mathsf{T}}$.

Prediction from an initial distribution

Suppose

$$\boldsymbol{x}_0 = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + 0.3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

After k years,

$$\boldsymbol{x}_k = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + 0.3(0.5)^k \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

At $k = 4$,

$$\boldsymbol{x}_4 = \begin{pmatrix} 0.61875 \\ 0.38125 \end{pmatrix}.$$

Application decision
The planning model predicts convergence toward 60% in North and 40% in South. The second mode halves each year, so the gap from equilibrium decreases rapidly.

Model limitation
Fixed transition percentages ignore changes in employment, housing, policy, climate, and population growth. The model supports scenario planning, not an unconditional forecast.

7. Python Laboratory

Compute eigenpairs, verify diagonalization, and simulate the transition model

```python
import numpy as np
import matplotlib.pyplot as plt

# Worked matrix
A = np.array([[4., 1.],
              [2., 3.]])
eigvals, P = np.linalg.eig(A)
D = np.diag(eigvals)
A_rebuilt = P @ D @ np.linalg.inv(P)
A4_spectral = P @ np.linalg.matrix_power(D, 4) @ np.linalg.inv(P)

print("Eigenvalues of A:", np.round(eigvals, 6))
print("Eigenvectors as columns:\n", np.round(P, 6))
print("A P - P D:\n", np.round(A @ P - P @ D, 10))
print("Reconstructed A:\n", np.round(A_rebuilt, 6))
print("A^4 from eigen-decomposition:\n", np.round(A4_spectral, 6))

# Two-region transition model
M = np.array([[0.8, 0.3],
              [0.2, 0.7]])
values, vectors = np.linalg.eig(M)

# Select the eigenvector whose eigenvalue is closest to 1
i = np.argmin(np.abs(values - 1))
steady = np.real(vectors[:, i])
steady = steady / steady.sum()

x = np.array([0.9, 0.1])
history = [x.copy()]
for year in range(1, 11):
    x = M @ x
    history.append(x.copy())
history = np.array(history)

print("Transition eigenvalues:", np.round(values, 6))
print("Steady distribution:", np.round(steady, 6))
print("Distribution after 4 years:", np.round(history[4], 6))

plt.plot(history[:, 0], "o-", label="North")
plt.plot(history[:, 1], "s-", label="South")
plt.axhline(steady[0], color="#1665D8", ls="--", alpha=0.6)
plt.axhline(steady[1], color="#0C8F8F", ls="--", alpha=0.6)
plt.xlabel("Year")
plt.ylabel("Population proportion")
plt.title("Convergence to the eigenvector with eigenvalue 1")
plt.legend()
plt.grid(alpha=0.25)
plt.tight_layout()
plt.show()
```

Selected output

```
Eigenvalues of A: [5. 2.]
A^4 from eigen-decomposition: [[422. 203.] [406. 219.]]
Transition eigenvalues: [1.  0.5]
Steady distribution: [0.6 0.4]
Distribution after 4 years: [0.61875 0.38125]
```

Numerical practice
`np.linalg.eig` returns eigenvectors as columns. Their order and scaling may differ from hand calculations; verify each pair using $\mathbf{A}v \approx \lambda v$ rather than comparing signs.

8. Concept Check and Practice

Common misconceptions

Misconception	Why it fails	Correct practice
The zero vector is an eigenvector	It has no direction and satisfies every homogeneous equation	Require $\boldsymbol{v} \neq \mathbf{0}$
Every matrix is diagonalizable	A matrix may lack enough independent eigenvectors	Compare geometric and algebraic multiplicities
Eigenvectors are unique	Any nonzero scalar multiple represents the same direction	Report an eigenspace or a convenient basis vector
NumPy must return the same vectors as hand work	Signs, scaling, and order can differ	Verify $\mathbf{A}\boldsymbol{v} \approx \lambda\boldsymbol{v}$

Three concept-check MCQs

1. If $\mathbf{A}\boldsymbol{v} = 3\boldsymbol{v}$ and $\boldsymbol{v} \neq \mathbf{0}$, then 3 is:

A. a determinant

B. an eigenvalue

C. a residual

D. a rank

2. A matrix is guaranteed to be diagonalizable when it has:

A. determinant zero

B. one repeated eigenvalue

C. n distinct eigenvalues

D. only zero entries

3. For a column-stochastic transition matrix, the steady-state eigenvalue is usually:

A. -1

B. 0

C. 1

D. 2

MCQ answers: 1-B, 2-C, 3-C

Practice exercises

1. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.
2. Find the eigenpairs of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and check orthogonality.
3. Verify that $(1,1)^{\mathsf{T}}$ is an eigenvector of $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
4. Decide whether $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.
5. Use the diagonalization on page 5 to calculate $\mathbf{A}^2$.
6. Find the spectral radius of $\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$.
7. Use the population model to calculate $\boldsymbol{x}_3$ from $\boldsymbol{x}_0 = (0.9, 0.1)^{\mathsf{T}}$.
8. Modify the Python code for $\boldsymbol{x}_0 = (0.2, 0.8)^{\mathsf{T}}$ and describe the convergence.

9. Scenario Challenge, Answers, and Takeaways

Scenario-based problem: service subscription movement

A platform groups users into Basic and Premium plans. Each month, 75% of Basic users remain Basic and 25% move to Premium; 80% of Premium users remain Premium and 20% move to Basic. The current distribution is $\boldsymbol{x}_0 = (0.8, 0.2)^\mathsf{T}$.

Using

$$\mathbf{M} = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix},$$

find the eigenvalues, long-run distribution, and predicted distribution after five months.

Brief answers to practice

Exercise	Brief answer
1	$\lambda = 3$ with $(1,0)^\mathsf{T}$; $\lambda = -1$ with $(0,1)^\mathsf{T}$
2	$\lambda = 3$ with $(1,1)^\mathsf{T}$; $\lambda = 1$ with $(1,-1)^\mathsf{T}$; dot product $= 0$
3	$\mathbf{A}(1,1)^\mathsf{T} = (4,4)^\mathsf{T} = 4(1,1)^\mathsf{T}$
4	No; $\lambda = 2$ has algebraic multiplicity 2 but only one independent eigenvector
5	$\mathbf{A}^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$
6	Eigenvalues are 1 and 0.5, so $\rho(\mathbf{A}) = 1$
7	$\boldsymbol{x}_3 = (0.6375, 0.3625)^\mathsf{T}$
8	The distribution approaches $(0.6, 0.4)^\mathsf{T}$ from the opposite side
Scenario	Eigenvalues 1 and 0.55; steady distribution $(4/9, 5/9)^\mathsf{T}$; $\boldsymbol{x}_5 \approx (0.4623, 0.5377)^\mathsf{T}$

Lecture summary

- Eigenvectors are invariant directions; eigenvalues measure scaling on those directions.
- The characteristic equation identifies eigenvalues, and null spaces identify eigenspaces.
- Diagonalization converts matrix powers into powers of diagonal entries.
- Symmetric matrices have real eigenvalues and orthogonal eigenvectors.
- Dominant eigenvalues govern long-term behaviour in many dynamic models.

Key terms: eigenvalue • eigenvector • characteristic polynomial • eigenspace • algebraic multiplicity • geometric multiplicity • diagonalization • spectral radius • steady state

Bridge to Lecture 9
Singular Value Decomposition extends eigenvector-style geometry to rectangular matrices and supports low-rank approximation, image compression, denoising, and principal component analysis.

Suggested references: Gilbert Strang, Introduction to Linear Algebra; David C. Lay, Steven R. Lay, and Judi J. McDonald, Linear Algebra and Its Applications; NumPy eigenvalue documentation.